Conformally invariant bending energy for hypersurfaces

2005 J. Phys. A: Math. Gen. 38 7943

(http://iopscience.iop.org/0305-4470/38/37/002)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **38** (2005) 7943–7955 doi:10.1088/0305-4470/38/37/002

Conformally invariant bending energy for hypersurfaces

Jemal Guven

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
Apdo Postal 70-543, 04510 México, DF, Mexico

Received 17 May 2005, in final form 19 July 2005
Published 31 August 2005
Online at stacks.iop.org/JPhysA/38/7943

Abstract
The most general conformally invariant bending energy of a closed four-dimensional surface, polynomial in the extrinsic curvature and its derivatives, is constructed. This invariance manifests itself as a set of constraints on the corresponding stress tensor. If the topology is fixed, there are three independent polynomial invariants: two of these are the straightforward quartic analogues of the quadratic Willmore energy for a two-dimensional surface; one is intrinsic (the Weyl invariant), the other extrinsic; the third invariant involves a sum of a quadratic in gradients of the extrinsic curvature—which is not itself invariant—and a quartic in the curvature. The four-dimensional energy quadratic in extrinsic curvature plays a central role in this construction.

PACS numbers: 04.60.Ds, 87.16.Dg, 46.70.Hg, 02.40.Hw

1. Introduction

The bending energy of a two-dimensional surface, quadratic in its extrinsic curvature, is invariant under scaling; size does not matter. What is less obvious is that this energy is also invariant under transformations of the three-dimensional background which preserve angles; it is conformally invariant. In particular, any two surface geometries related to each other by inversion in a point have the same energy. This property was first studied systematically by Willmore in the 1960s [1]. More recently, it was discovered to lie at the heart of some fascinating connections between differential geometry and integrable systems [2]. In the 1970s it was recognized that the physics of a fluid membrane—formed by the spontaneous aggregration of amphiphilic molecules into bilayers in water—is captured completely at mesoscopic scales by geometrical degrees of freedom [3, 4]. On such scales the membrane can be modelled as a two-dimensional surface; at lowest order, the principal cost in energy is associated with bending this surface [5]. Remarkably, all of the molecular details get telescoped into a single rigidity modulus. A role was also found for a relativistic counterpart in the 1980s: an addition quadratic in extrinsic curvature to the action of a relativistic string accounts for the behaviour of colour flux tubes in QCD [6–8].

Higher-dimensional analogues of the two-dimensional bending energy are also of potential interest both as statistical field theories and relativistically as braneworld actions. If the surface dimension differs from two, however, the energy quadratic in extrinsic curvature is no longer scale invariant, much less conformally invariant: a higher-dimensional sphere without a constraint on its area will collapse; tension is necessarily introduced. Conformal invariant energies, polynomial in the extrinsic curvature, are however simple to construct: the building block is the traceless part of the extrinsic curvature tensor which transforms by a multiplicative factor [1, 9]. In four dimensions, for a hypersurface of fixed topology, there are two independent conformal invariant energies quartic in the extrinsic curvature. The snag is that these invariants alone cannot accurately describe a conformally invariant theory of bending. The reason is simple: when expanded as a power series in terms of a height function, they begin with a term quartic in this function. Thus, they vanish in the Gaussian approximation to the energy truncating it at the quadratic in the height function. In particular, there is no harmonic regime to describe fluctuations about a flat geometry. On dimensional grounds, the relevant invariant must involve curvature gradients. This invariant will play a role in the formulation of a consistent statistical field theory of four-dimensional hypersurfaces. Its identification, however, is somewhat less trivial than that of its quartic counterparts. To do this, it will be useful to approach the problem from a global point of view which focuses directly on the transformation properties of the surface energy rather than the individual tensors which appear within it. For the sake of simplicity, we will focus on a closed hypersurface of fixed topology; more simple still, think topological 4-sphere.

Consider any energy, constructed using the metric and the extrinsic curvature, which is invariant under reparametrizations of the hypersurface and Euclidean motions of space. If the theory is invariant under translations the response of this energy to an arbitrary infinitesimal deformation of the hypersurface can be expressed in terms of a stress tensor [10, 11]. In [11] it was shown that the tangential stress f^{ab} has two contributions: one of these is the metric stress tensor T^{ab} which determines the response of the energy to changes in the intrinsic geometry; the second, which determines the response to changes in the extrinsic geometry, involves the functional derivative of the energy with respect to the extrinsic curvature, $\mathcal{H}^{ab}$. One is interested, in particular, in determining the response of the energy to the deformation of the hypersurface induced by an infinitesimal conformal transformation. It is possible to characterize this response in a remarkably succinct way in terms of traces: that of f^{ab} and that of $\mathcal{H}^{ab}$. Conformal invariance will place constraints on these traces. In contrast to a conformal invariant of the intrinsic geometry which has a representation with $\mathcal{H}^{ab} = 0$, these constraints may be satisfied in a very subtle way by an invariant of the the extrinsic geometry.

The task is to identify energies that are consistent with these constraints. While the focus will be on closed four-dimensional hypersurfaces, the techniques developed will be independent of the dimension. We first briefly describe the construction, within this framework, of the two well-known four-dimensional conformally invariant energies quartic in extrinsic curvature. Modulo the Gauss–Codazzi equations, which identify the intrinsic Riemann tensor with a quadratic in the extrinsic curvature, one of these invariants is the Weyl invariant associated with the intrinsic geometry of the hypersurface, and thus insensitive to the particular way the hypersurface is embedded. The third invariant involves a balance of a part quadratic in curvature gradients with a quartic in curvature; neither term on its own is conformally invariant. We show how this constraint can be satisfied by tuning the quartic so that the two trace terms cancel. Intriguingly, this cancellation involves properties of the four-dimensional

Willmore energy quadratic in the extrinsic curvature (which is not itself a conformal invariant in this dimension) in an essential way.

2. Linear response, the Euler–Lagrange derivative as a divergence, and the stress

Consider a closed D-dimensional hypersurface embedded in R^{D+1}. This hypersurface is described locally by the embedding, $\mathbf{x} = \mathbf{X}(\xi^a)$. Here $\mathbf{x} = (x^1, \ldots, x^{D+1})$ and $\xi^a, a = 1, \ldots, D$ parametrize the hypersurface. The metric tensor and extrinsic curvature induced by $\mathbf{X}$ are respectively $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$ and $K_{ab} = \mathbf{e}_a \cdot \partial_b \mathbf{n}$, where $\mathbf{e}_a = \partial_a \mathbf{X}, a = 1, \ldots, D$ are tangent and $\mathbf{n}$ is the unit normal. The Gauss–Weingarten equations are $\nabla_a \mathbf{e}_b = -K_{ab}\mathbf{n}$ and $\partial_a \mathbf{n} = K_a{}^b \mathbf{e}_b$ [12]. ∇_a is the covariant derivative compatible with g_{ab}; spatial indices get raised with the inverse metric g^{ab}. We are interested in functionals of $\mathbf{X}$ which are invariant under reparametrizations of the hypersurface.

The metric and extrinsic curvature are both invariant under the change in $\mathbf{X}$ induced by a Euclidean motion in R^{D+1}: a Euclidean invariant energy $H[\mathbf{X}]$ can therefore be cast as a functional of the metric, the extrinsic curvature and its derivatives,

$$H[\mathbf{X}] = \int \mathrm{d}A\, \mathcal{H}(g_{ab}, K_{ab}, \nabla_a K_{bc}, \ldots). \tag{1}$$

The area element induced on the hypersurface is $\mathrm{d}A = \sqrt{\det g_{ab}}\, d^D\xi$. We wish, in particular, to construct an energy which is invariant under deformations induced by a conformal change of the Euclidean background:

$$\delta\mathbf{x} = \mathbf{a} + \mathbf{B}\mathbf{x} + \lambda\mathbf{x} + \mathbf{x}^2\mathbf{c} - 2(\mathbf{c}\cdot\mathbf{x})\mathbf{x}, \tag{2}$$

where $\mathbf{a}$ and $\mathbf{c}$ are two constant vectors ($\mathbf{c}$ has dimensions of inverse length), $\mathbf{B}$ is an antisymmetric $(D+1) \times (D+1)$ matrix, and λ is a positive constant. $\delta\mathbf{x}$ is the sum of an infinitesimal Euclidean motion, a change of scale and a special conformal transformation. The latter exponentiates to the composition of an inversion $\mathbf{x} \to \mathbf{x}/\mathbf{x}^2$, a translation through the vector $\mathbf{c}$, and another inversion, $\mathbf{x} \to (\mathbf{x} + \mathbf{x}^2\mathbf{c})/(1 + 2\mathbf{c}\cdot\mathbf{x} + \mathbf{c}^2\mathbf{x}^2)$. Both g_{ab} and K_{ab} are invariants under Euclidean motions: as a result, any energy of the form (1) will also be by construction. What one now needs to do is characterize the constraints placed on H by invariance under scaling and special conformal transformations.

It is useful to first determine the linear response of H to any small deformation of the hypersurface. This task is simplified by exploiting invariance under Euclidean motions of the ambient space. While Noether's theorem informs us that the Euler–Lagrange derivative can always be cast as the divergence of a stress tensor, in all but the simplest case—an energy proportional to the area functional describing surface tension—the identification of this tensor is subtle: unlike the stress associated with area, the stress will depend not only on the intrinsic geometry but also on how the hypersurface bends. The tug on the hypersurface will possess a normal component.

A small deformation of the hypersurface is described by the infinitesimal change in the embedding functions $\mathbf{X}$

$$\mathbf{X} \to \mathbf{X} + \delta\mathbf{X}. \tag{3}$$

Note the following points:

(1) As a consquence of the reparametrization invariance of H in a closed geometry, the response of H is independent of the tangential projection of $\delta\mathbf{X}$; thus

$$\delta H = \int \mathrm{d}A\, \mathcal{E}\mathbf{n}\cdot\delta\mathbf{X} \tag{4}$$

involves only the normal projection. The Euler–Lagrange derivative of H with respect to $\mathbf{X}$ is denoted by $\mathcal{E}\mathbf{n}$.

(2) The translational invariance of H implies that its Euler–Lagrange derivative is a divergence [10, 11]

$$\mathcal{E}\mathbf{n} = \nabla_a \mathbf{f}^a. \tag{5}$$

The hypersurface current $\mathbf{f}^a \cdot \mathbf{a}$ is associated with the invariance of H under a translation $\delta\mathbf{X} = \mathbf{a}$. When the Euler–Lagrange equation $\mathcal{E} = 0$ is satisfied, this current is conserved. The closure of the geometry then permits δH to be recast in the remarkably simple form

$$\delta H = -\int \mathrm{d}A\, \mathbf{f}^a \cdot \nabla_a \delta\mathbf{X}. \tag{6}$$

This expression involves one less derivative than equation (4). Note that one does not need to know how $\mathbf{f}^a$ itself transforms. Equation (6) is valid whether or not the Euler–Lagrange equation is satisfied. This equation will be used to examine the response of H to the deformation in the hypersurface induced by conformal transformations of space.

3. The stress

The stress $\mathbf{f}^a$ associated with H is given by

$$\mathbf{f}^a = \left(T^{ab} - \mathcal{H}^{ac} K_c{}^b\right)\mathbf{e}_b - \nabla_b \mathcal{H}^{ab}\mathbf{n}, \tag{7}$$

where $\mathcal{H}^{ab}$ is the functional derivative of H with respect to K_{ab},

$$\mathcal{H}^{ab} = \frac{\partial \mathcal{H}}{\partial K_{ab}} - \nabla_c \left(\frac{\partial \mathcal{H}}{\partial \nabla_c K_{ab}}\right) + \cdots \tag{8}$$

and $T^{ab} = -(2/\sqrt{g})\delta H/\delta g_{ab}$ is the intrinsic stress tensor associated with the metric g_{ab}. This construction involves treating g_{ab} and K_{ab} as independent variables in $\mathcal{H}$; to do this consistently requires one to introduce a set of auxiliary variables to constrain g_{ab} and K_{ab} to satisfy the Gauss–Weingarten structural relationships. The ellipsis appearing on the rhs of equation (8) indicates terms which appear if $\mathcal{H}$ depends on derivatives of K_{ab} higher than first. A simple derivation of equation (7) is provided in [11]. We note, in particular, that $\mathbf{f}^a$ decomposes into tangential and normal parts

$$\mathbf{f}^a = f^{ab}\mathbf{e}_b + f^a\mathbf{n}. \tag{9}$$

This decomposition has the following properties which are relevant:

(1) The tangential projections of equation (5) provide a consistency condition on the components of the stress

$$\nabla_a f^{ab} + K^{ab} f_a = 0; \tag{10}$$

the normal component determines $\mathcal{E}$:

$$\mathcal{E} = \nabla_a f^a - K_{ab} f^{ab}. \tag{11}$$

(2) The normal stress f^a is always a divergence.

(3) Even though both $\mathcal{H}^{ab}$ and T^{ab} are symmetric tensors, f^{ab} will not generally be symmetric. On one hand, as equation (11) indicates clearly, only the symmetric part of f^{ab} contributes to the Euler–Lagrange derivative. An anti-symmetric contribution, if present, will however show up in the consistency conditions (10) and so cannot be discarded naively.

Let us now consider specific forms for the function $\mathcal{H}$ appearing in equation (1) which will be used in the construction of conformal invariants.

3.1. $\mathcal{H}(g_{ab}, K_{ab})$

Suppose that $\mathcal{H}$ does not involve derivatives of K_{ab}: $\mathcal{H} = \mathcal{H}(g_{ab}, K_{ab})$. Then

$$\mathcal{H}^{ab} = \partial\mathcal{H}/\partial K_{ab}, \tag{12}$$

and it is simple to show that

$$T^{ab} = 2\mathcal{H}^{ac}K_c{}^b - \mathcal{H}g^{ab}, \tag{13}$$

so that

$$\mathbf{f}^a = (\mathcal{H}^{ac}K_c{}^b - \mathcal{H}g^{ab})\mathbf{e}_b - \nabla_b\mathcal{H}^{ab}\mathbf{n}. \tag{14}$$

$\mathcal{H}^{ab}$ is a symmetric polynomial in K^{ab}. Thus f^{ab} is a symmetric tensor. A straightforward calculation gives

$$\mathcal{E} = -\nabla_a\nabla_b\mathcal{H}^{ab} - K_{ac}K^c{}_b\mathcal{H}^{ab} + \mathcal{H}K, \tag{15}$$

where $K = g^{ab}K_{ab}$ is the trace of K_{ab} (D times the mean curvature).

In particular, for the Canham–Helfrich or Willmore energy, one has

$$H_0 = \frac{1}{2}\int \mathrm{d}A\, K^2, \tag{16}$$

$\mathcal{H} = K^2/2$ and $\mathcal{H}^{ab} = Kg^{ab}$, so that

$$\mathbf{f}^a = K\big(K^{ab} - \tfrac{1}{2}g^{ab}K\big)\mathbf{e}_b - \nabla_a K\mathbf{n}, \tag{17}$$

and [1] (see also [13])

$$\mathcal{E} = -\nabla^2 K + \tfrac{1}{2}K(K^2 - 2K^{ab}K_{ab}). \tag{18}$$

If $K = 0$, then $\mathcal{E} = 0$. The relationship between $\mathcal{E}$ and $\mathbf{f}^a$ for H_0 will play a role in the construction of a higher-derivative conformal invariant of a four-dimensional hypersurface.

Note that it is unnecessary to admit an explicit intrinsic curvature dependence in $\mathcal{H}$. This is because the Gauss–Codazzi equations [12]

$$\mathcal{R}_{abcd} = K_{ac}K_{bd} - K_{ad}K_{bc} \tag{19}$$

completely fix the Riemann tensor, as well as its contractions, the Ricci tensor $\mathcal{R}_{ab} = g^{cd}R_{acbd}$ and the scalar curvature $\mathcal{R} = g^{ab}\mathcal{R}_{ab}$, in terms of the extrinsic curvature[1]. However, if one is interested explicitly in a functional of the intrinsic geometry, $\mathcal{H} = \mathcal{H}(g_{ab}, \mathcal{R}_{abcd}, \nabla_e\mathcal{R}_{abcd}, \ldots)$, it may then be more appropriate to treat these tensors as functionals of g_{ab} alone, and ignore the integrability conditions (19). If this is done, $\mathcal{H}^{ab} = 0$ and T^{ab} is the stress tensor of the (purely) metric theory defined by $\mathcal{H}$. Now, $f^{ab} = T^{ab}$ and it is manifestly symmetric; furthermore $f^a = 0$.[2] The Euler–Lagrange derivative is then simply $\mathcal{E} = -K_{ab}T^{ab}$; the consistency condition then reads $\nabla_a T^{ab} = 0$—the metric stress tensor is conserved. Clearly, it does not matter how one decides to split the burden on g_{ab} and K_{ab}, so long as it is done consistently when performing the variations in the derivation of $\mathbf{f}^a$. As discussed in detail elsewhere, if $\mathbf{f}^a$ is treated as a differential form, the difference between its values in the two representations is an exact form.

[1] The Riemann tensor is defined intrinsically by the failure of the ∇_a to commute: for a space vector V_a, we have the Ricci identity $(\nabla_a\nabla_b - \nabla_b\nabla_a)V_c = \mathcal{R}_{abc}{}^d V_d$.

[2] Intrinsically defined invariants are not the only ones possessing this property. One can show that the geometrical invariants constructed out of the symmetric polynomials in the curvature, $P_N(\sigma_1, \ldots, \sigma_D)$, $N \leqslant D$, where $\{\sigma_i\}$ are the principal curvatures, do also. For example, when $N = 1$, $P_1 = K$ and $f^{ab} = K^{ab} - g^{ab}K$, which is conserved, so that $f^a = 0$.

3.2. $\mathcal{H}(g_{ab}, K_{ab}, \nabla_c K_{ab})$

If one extends the class of functionals to include a dependence on $\nabla_c K_{ab}$, there are few useful general statements concerning the structure of $\mathbf{f}^a$. Our limited goal, however, is to identify conformal invariants of closed hypersurfaces so we do not need to consider the most general form.

Consider candidate polynomials in $\nabla_a K_{bc}$ and K_{ab} that are consistent with scale invariance. When $D = 2$, there are none. When $D = 3$, there is a Chern–Simons type topological energy; it vanishes on a closed geometry. When $D = 4$, the quadratics in $\nabla_c K_{ab}$ are scale invariant. As shown in [14], however, any quadratic in derivatives of K_{ab} is expressible, modulo a divergence, as a sum of the simple invariant

$$H_1 = \frac{1}{2}\int \mathrm{d}A(\nabla K)^2 \tag{20}$$

and an integral over some quartic in K_{ab}. The latter is of the form, $\mathcal{H}(g_{ab}, K_{ab})$, already considered in section 3.1. So H_1 is the only invariant that needs to be considered.

The demonstration of this claim involves the Codazzi–Mainardi integrability conditions

$$\nabla_a K_{bc} - \nabla_b K_{ac} = 0, \tag{21}$$

as well as the Ricci identity applied to K_{ab}

$$[\nabla_a, \nabla_b]K_{cd} = R_{abc}{}^f K_{fd} + R_{abd}{}^f K_{cf}. \tag{22}$$

Consider the energy constructed using the quadratic $\nabla_a K_{bc}\nabla^a K^{bc}$. One first uses equation (21) followed by an integration by parts to obtain

$$\int \mathrm{d}A(\nabla_a K_{bc})(\nabla^a K^{bc}) = \int \mathrm{d}A(\nabla_a K_{bc})(\nabla^b K^{ac}) = -\int \mathrm{d}A\, K^b{}_c \nabla_a \nabla_b K^{ac}. \tag{23}$$

One then makes use of equation (22) to switch derivatives so that

$$\int \mathrm{d}A(\nabla_a K_{bc})(\nabla^a K^{bc}) = -\int \mathrm{d}A\big(K_c{}^b \nabla_b \nabla_a K^{ac} - \mathcal{R}_{abcd} K^{ac} K^{bd} + \mathcal{R}_{ab} K^{ac} K_c{}^b\big). \tag{24}$$

The contracted Codazzi–Mainardi equations, $\nabla_a K^{ab} - \nabla^b K = 0$, and another integration by parts are applied to the first term to nudge it into the required form:

$$\int \mathrm{d}A\, K_c{}^b \nabla_b \nabla_a K^{ac} = -\int \mathrm{d}A\, \nabla_b K^b{}_c \nabla_a K^{ac} = -\int \mathrm{d}A\, \nabla^c K \nabla_c K. \tag{25}$$

One concludes that

$$\begin{aligned}\int \mathrm{d}A(\nabla_a K_{bc})(\nabla^a K^{bc}) &= \int \mathrm{d}A(\nabla^c K \nabla_c K + \mathcal{R}_{abcd} K^{ac} K^{bd} - \mathcal{R}_{ab} K^{ac} K_c{}^b)\\ &= \int \mathrm{d}A(\nabla^c K \nabla_c K + (\operatorname{tr} K^2)^2 - K \operatorname{tr} K^3).\end{aligned} \tag{26}$$

The notation $\operatorname{tr} K^n = K_{a_1}{}^{a_2} \cdots K_{a_n}{}^{a_1}$ has been introduced. On the second line, the Gauss–Codazzi equations (19) have been used to eliminate the Riemann tensor in favour of a quadratic in extrinsic curvature. The energy H_1, given by equation (20), is reproduced modulo a quartic in extrinsic curvature.

Note that for H_1 one has (this is true for any dimension D)

$$\mathcal{H}^{ab} = -\nabla_c(g^{ab}\nabla^c K) = -g^{ab}\nabla^2 K \tag{27}$$

and

$$T^{ab} = \nabla^a K \nabla^b K - \tfrac{1}{2} g^{ab}(\nabla K)^2 - 2K^{ab}\nabla^2 K. \tag{28}$$

The second derivative term originates in the variation of $\nabla_a K$ with respect to g_{ab}. The correponding stress tensor is [15]

$$\mathbf{f}^a = \big[\nabla^a K \nabla^b K - \tfrac{1}{2} g^{ab} (\nabla K)^2 - K^{ab} \nabla^2 K\big] \mathbf{e}_b + \nabla^a \nabla^2 K \mathbf{n}. \tag{29}$$

Again f^{ab} is symmetric[3]. In this case, it is simple to check that

$$\mathcal{E} = [\nabla^2 + \operatorname{tr} K^2] \nabla^2 K - K^{ab} \big[\nabla_a K \nabla_b K - \tfrac{1}{2} g_{ab} (\nabla K)^2\big]. \tag{30}$$

Note that if $K = 0$, then $\mathcal{E} = 0$ so that minimal hypersurfaces also minimize H_1.

4. Scaling

It is trivial to identify energies which are scale invariant. However, the imprint of scale invariance on the stress tensor is subtle and it will be relevant to our interpretation of the response, under special conformal transformations, of the energy in terms of the stress tensor.

Consider an energy with a fixed scaling dimension. Under a change of scale $\mathbf{X} \to \Lambda \mathbf{X}$, where Λ is a positive constant, one has

$$H[\Lambda \mathbf{X}] = \Lambda^{D+d} H[\mathbf{X}] \tag{31}$$

for some d, or alternatively, in terms of the corresponding density, $\mathcal{H}[\Lambda \mathbf{X}] = \Lambda^d \mathcal{H}[\mathbf{X}]$. H is scale invariant when $d = -D$.

Consider now an infinitesimal change of scale, $\Lambda = 1 + \lambda$; at first order in λ, equation (31) gives

$$\delta_\lambda H = (D + d) \lambda H. \tag{32}$$

On the other hand, the first order variation equation (6) with the substitution $\delta \mathbf{X} = \lambda \mathbf{X}$ expresses $\delta_\lambda H$ in terms of the trace of the tangential stress,

$$\delta_\lambda H = -\lambda \int \mathrm{d}A \, f^a{}_a, \tag{33}$$

where $f^a{}_a = g_{ab} f^{ab}$. Comparison of equation (32) with equation (33) furnishes an identity,

$$(D + d) H = -\int \mathrm{d}A \, f^a{}_a. \tag{34}$$

Only the trace of the tangential stress tensor contributes to the change of H under scaling. Locally, this implies that

$$f^a{}_a = -(D + d) \mathcal{H} + \nabla_a G^a, \tag{35}$$

where G^a is a hypersurface vector field. Modulo a divergence, the trace is proportional to the integrand.

For functionals of the form $\mathcal{H}(g_{ab}, K_{ab})$ one can show that $G^a = 0$.[4] In particular, a scale invariant functional of this form has vanishing tangential trace: $f^a{}_a = 0$. This is *not* true

[3] Evidently, one has to proceed to a relatively high order energy to produce an f^{ab} which is not symmetric: for $\mathcal{H} = K^{ab} \nabla_a K \nabla_b K$, $\mathcal{H}^{ab}$ does not commute with K_{ab} and thus f^{ab} is not symmetric.

[4] To see this, consider the response to a deformation of H on a region with boundary: For functionals of the form $\mathcal{H}(g_{ab}, K_{ab})$, equation (6) is replaced by

$$\delta H = -\int \mathrm{d}A \, \mathbf{f}^a \cdot \nabla_a \delta \mathbf{X} + \int \mathrm{d}A \, \nabla_a [\mathcal{H}^{ab} \mathbf{e}_b \cdot \delta \mathbf{n}]. \tag{36}$$

In particular, under a change of scale, $\delta \mathbf{n} = 0$ and equation (36) reproduces equation (33) without any boundary term.

of higher derivative scale invariants: equation (34) does, however, imply that the trace is the divergence of a hypersurface vector field:

$$f^a{}_a = \nabla_a G^a. \tag{37}$$

In particular, in the case of the functional H_1 defined by equation (20), inspection of equation (29) gives for the trace of f_{ab},

$$f^a{}_a = \frac{4-D}{2}(\nabla K)^2 - \nabla^2 K^2/2. \tag{38}$$

One identifies $G^a = -\nabla^a K^2/2$.

5. Special conformal transformations

Infinitesimally, a special conformal transformation induces a change in $\mathbf{X}$ given by

$$\delta_{\mathbf{c}}\mathbf{X} = \mathbf{X}^2\mathbf{c} - 2(\mathbf{c}\cdot\mathbf{X})\mathbf{X}. \tag{39}$$

The corresponding response of the energy is determined using equation (6):

$$\begin{aligned}\delta_{\mathbf{c}}H &= -\int \mathrm{d}A\, \mathbf{f}^a \cdot \nabla_a \delta_{\mathbf{c}}\mathbf{X} = -\int \mathrm{d}A[f^{ab}(\mathbf{e}_b\cdot\nabla_a\delta\mathbf{X}) + f^a(\mathbf{n}\cdot\nabla_a\delta\mathbf{X})] \\ &= 2\int \mathrm{d}A\left[f^a{}_a(\mathbf{c}\cdot\mathbf{X}) - f^a\mathbf{c}\cdot\mathbf{f}_{0a}\right] + 2\int \mathrm{d}A\, f^{ab}\left[(\mathbf{e}_a\cdot\mathbf{c})(\mathbf{e}_b\cdot\mathbf{X}) - (a\leftrightarrow b)\right],\end{aligned} \tag{40}$$

where

$$\mathbf{f}_0^a = (\mathbf{e}^a\cdot\mathbf{X})\mathbf{n} - (\mathbf{n}\cdot\mathbf{X})\mathbf{e}^a. \tag{41}$$

The identities

$$\mathbf{e}_a\cdot\nabla_b\delta_{\mathbf{c}}\mathbf{X} = 2\left[(\mathbf{e}_a\cdot\mathbf{c})(\mathbf{e}_b\cdot\mathbf{X}) - (a\leftrightarrow b)\right] - 2(\mathbf{c}\cdot\mathbf{X})g_{ab} \tag{42}$$

and

$$\mathbf{n}\cdot\nabla_a\delta_{\mathbf{c}}\mathbf{X} = 2\left[(\mathbf{e}_a\cdot\mathbf{X})(\mathbf{c}\cdot\mathbf{n}) - (\mathbf{e}_a\cdot\mathbf{c})(\mathbf{X}\cdot\mathbf{n})\right] = 2\mathbf{c}\cdot\mathbf{f}_{0a} \tag{43}$$

have been used on the second line of equation (40). In the case of the energies we will consider f^{ab} is symmetric so that the term appearing on the last line in equation (40) vanishes. The equation thus simplifies to

$$\delta_{\mathbf{c}}H = 2\int \mathrm{d}A\left[f^a{}_a(\mathbf{c}\cdot\mathbf{X}) - f^a\mathbf{c}\cdot\mathbf{f}_{0a}\right]. \tag{44}$$

Further simplification is possible using the structure of $\mathbf{f}^a$. Using the fact that $f^a = -\nabla_b\mathcal{H}^{ab}$, where $\mathcal{H}^{ab}$ is given by equation (8), the second term appearing on the rhs of equation (44) can be cast as

$$\int \mathrm{d}A\, f^a\mathbf{c}\cdot\mathbf{f}_{0a} = \int \mathrm{d}A\,\mathcal{H}^{ab}\mathbf{c}\cdot\nabla_b\mathbf{f}_{0a}. \tag{45}$$

However, the definition of $\mathbf{f}_0^a$ (41) gives

$$\nabla_b\mathbf{f}_{0a} = g_{ab}\mathbf{n} + K_b{}^c((\mathbf{X}\cdot\mathbf{e}_a)\mathbf{e}_c - (a\leftrightarrow c)). \tag{46}$$

The tangential projection is anti-symmetric in a and b and so does not contribute to the rhs of equation (45) if $f^{ab} = T^{ab} - \mathcal{H}^{ac}K_c{}^b$ is symmetric. Even when it is not, it cancels against an identical term appearing on the last line in equation (40). It then follows that:

$$\delta_{\mathbf{c}}H = 2\int \mathrm{d}A\left[f^a{}_a(\mathbf{c}\cdot\mathbf{X}) - \mathcal{H}^a{}_a(\mathbf{c}\cdot\mathbf{n})\right], \tag{47}$$

where $\mathcal{H}^a{}_a = g_{ab}\mathcal{H}^{ab}$. The response of H to an infinitesinal special conformal transformation has been expressed as a difference of two terms. Each of these terms involves a trace. Note that in the case of any intrinsic geometrical invariant, there exists a representation in which the second term vanishes.

The energy H is conformally invariant if and only if equations (37) and

$$f^a{}_a(\mathbf{c}\cdot\mathbf{X}) - \mathcal{H}^a{}_a(\mathbf{c}\cdot\mathbf{n}) = \nabla_a h^a \tag{48}$$

are satisfied. h^a is a hypersurface vector field.

We now construct a four-dimensional energy, polynomial in the curvature and its derivatives, that is consistent with these constraints.

6. Conformally invariants polynomial in K_{ab}

A scale invariant energy with a density $\mathcal{H}$ depending on g_{ab} and K_{ab} but not on their derivatives has traceless f^{ab}: $f^a{}_a = 0$. To be invariant under special conformal transformations, one also requires that

$$\int \mathrm{d}A\, \mathcal{H}^a{}_a\mathbf{n} = 0. \tag{49}$$

This is clearly satisfied if $\mathcal{H}^{ab}$ is also traceless: $\mathcal{H}^a{}_a = 0$. It is straightforward to construct polynomial functionals with this property. These are the well-known invariants involving the traceless part of the extrinsic curvature tensor $\tilde{K}_{ab}$ $\left(\tilde{K}^a{}_a = 0\right)$

$$\tilde{K}_{ab} = K_{ab} - \frac{K}{D}g_{ab}. \tag{50}$$

Let $\mathcal{H}$ be a product of terms $\mathcal{H}_n$, each of which is a trace over a product of n $\tilde{K}_{ab}$s (see definition below equation (26)); $\mathcal{H}_n = \operatorname{tr}\tilde{K}^n$. Note that

$$\Pi_{ab}{}^{cd} := \frac{\partial\tilde{K}_{ab}}{\partial K_{cd}} = \frac{1}{2}\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right) - \frac{1}{D}g_{ab}g^{cd} \tag{51}$$

projects out the trace $\left(\Pi_{ab}{}^{cd}g_{cd} = 0\right)$. For each factor $\mathcal{H}_n$, we thus find that

$$g_{ab}\frac{\partial\mathcal{H}_n}{\partial K_{ab}} = 0. \tag{52}$$

Consequently, $\mathcal{H}^a{}_a = 0$ [1] (see also [16]). Two-dimensional surfaces are considered in an appendix.

In four dimensions, there are two polynomial conformal invariants constructed this way corresponding to the two independent quartics $\operatorname{tr}\tilde{K}^4$ and $(\operatorname{tr}\tilde{K}^2)^2$. One linear combination of the two is the Weyl invariant of the intrinsic geometry [17].

It is possible to satisfy equation (49) in a rather less trivial way. It was seen that the translation invariance of any functional of the form (1) implies the identity equation (5) between its Euler–Lagrange derivative and the hypersurface divergence of a stress tensor. If this identity is integrated over a closed hypersurface it follows immediately that the Euler–Lagrange derivative satisfies

$$\int \mathrm{d}A\, \mathcal{E}\mathbf{n} = 0. \tag{53}$$

Thus, if it is possible to cast $\mathcal{H}^a{}_a$ as the Euler–Lagrange derivative of some translationally invariant functional of the form (1), then equation (49) will be satisfied. However, one can show that the only energy densities constructed using K_{ab} consistent with this condition are proportional to a sum of the symmetric polynomials in the principal curvatures. Consistency with scale invariance leaves only the determinant. Thus the only conformal invariant generated this way is the Gauss–Bonnet topological invariant [12].

Note that the Paneitz invariant, which has been the centre of recent research, is the difference between the Gauss–Bonnet and the Weyl invariants [18, 19]. As such, it is an invariant of the intrinsic geometry.

7. Conformal invariant quadratic in gradients of K_{ab}

In four dimensions, the functional H_1 defined by equation (20) is scale invariant but it is not conformally invariant. It is possible, however, to construct a conformal invariant by adding to H_1 the integral of an appropriate quartic in K_{ab}. One way to identify this quartic is as follows:

(i) First determine how H_1 transforms:

for H_1, neither $f^a{}_a$ nor $\mathcal{H}^a{}_a$ vanishes; one must contend with the two terms appearing in equation (47). For H_1, equation (38) gives $f^a{}_a = -\nabla^2 K^2/2$. On performing two integrations by parts and using the Gauss–Weingarten equations, one finds that

$$\int \mathrm{d}A\, f^a{}_a(\mathbf{c}\cdot\mathbf{X}) = \frac{1}{2}\int \mathrm{d}A\, K^3(\mathbf{c}\cdot\mathbf{n}). \tag{54}$$

In addition, using $\mathcal{H}^{ab} = -g^{ab}\nabla^2 K$,

$$\int \mathrm{d}A\, \mathcal{H}^a{}_a(\mathbf{c}\cdot\mathbf{n}) = -4\int \mathrm{d}A\, \nabla^2 K(\mathbf{n}\cdot\mathbf{c}). \tag{55}$$

The identities (54) and (55) are now substituted into equation (47) to give for $\delta_{\mathbf{c}} H_1$:

$$\delta_{\mathbf{c}} H_1 = \int \mathrm{d}A(K^3 + 8\nabla^2 K)(\mathbf{n}\cdot\mathbf{c}). \tag{56}$$

(ii) Next note that the rhs of equation (56) can be simplified by using an identity associated with the quadratic energy H_0 defined by equation (16). Using equation (18), equation (5) implies that

$$\int \mathrm{d}A\left[\nabla^2 K + \left(\mathrm{tr}\, K^2 - \frac{1}{2}K^2\right)K\right](\mathbf{n}\cdot\mathbf{c}) = 0, \tag{57}$$

for any $\mathbf{c}$. This identity allows $\delta_{\mathbf{c}} H_1$ given by equation (56) to be expresssed in terms of a cubic polynomial in K_{ab}:

$$\delta_{\mathbf{c}} H_1 = -8\int \mathrm{d}A\left(\mathrm{tr}\, K^2 - \frac{5}{8}K^2\right)K(\mathbf{n}\cdot\mathbf{c}). \tag{58}$$

Now let $\mathcal{H}_2 = (\nabla K)^2/2 + \mathcal{H}'$ where $\mathcal{H}'$ is quartic in K_{ab}. If

$$g_{ab}\frac{\partial\mathcal{H}'}{\partial K_{ab}} = 8\left(\mathrm{tr}\, K^2 - \frac{5}{8}K^2\right)K, \tag{59}$$

then $H = \int \mathrm{d}A\mathcal{H}$ will be conformally invariant. The choice of $\mathcal{H}'$ is clearly not unique, however, it is modulo a linear combination of the two conformally covariant quartics, $\mathrm{tr}\,\tilde{K}^4$ and $(\mathrm{tr}\,\tilde{K}^2)^2$. The simplest choice is a linear combination of the two invariants, K^4 and $K^2\,\mathrm{tr}\,K^2$. A short calculation gives

$$\mathcal{H}' = K^2\,\mathrm{tr}\,K^2 - \tfrac{7}{16}K^4 \tag{60}$$

and one identified the following four-dimensional conformally invariant energy:

$$H_2 = \frac{1}{2}\int \mathrm{d}A\left((\nabla K)^2 - \frac{7}{8}K^4 + 2K^2\,\mathrm{tr}\,K^2\right). \tag{61}$$

This identification appears to be new. Unlike the two quartic conformal invariants in four dimensions, the conformal invariance of H_2 involves a delicate balance between gradients and quartics.

With our choice of quartic $\mathcal{H}'$, H_2 is not positive, unlike the invariants constructed using $\operatorname{tr}\tilde{K}^4$ and $(\operatorname{tr}\tilde{K}^2)^2$, which are. It is, however, possible to form a positive conformally invariant gradient energy by adding an appropriate linear combination of the other invariants. The most general four-dimensional conformally invariant energy polynomial in the extrinsic curvature will involve a linear combination of all three invariants.

Note that a physically realistic four-dimensional generalization of the Willmore energy will involve H_2. Consider the Monge description of the hypersurface in terms of a height function h above a reference plane. With respect to Cartesian coordinates on this plane, the extrinsic curvature tensor takes the form

$$K_{ab} = -\frac{\nabla_a \nabla_b h}{(1 + (\nabla h)^2)^{1/2}}, \tag{62}$$

where now ∇_a is the flat derivative on this plane. To lowest order in h, $K_{ab} \approx -\nabla_a \nabla_b h + \mathcal{O}(h^3)$. In the Gaussian approximation, quadratic in h, all quartics in K_{ab} vanish; in particular, the conformal invariants constructed using $\operatorname{tr}\tilde{K}^4$ and $(\operatorname{tr}\tilde{K}^2)^2$ both vanish. For H_2 given by equation (61) only the gradient term survives and one is left with

$$H_2 = \frac{1}{2}\int \mathrm{d}A_\perp (\nabla \Delta h)^2 + \mathcal{O}(h^4), \tag{63}$$

where $\mathrm{d}A_\perp$ is the area element and Δ is the flat Laplacian on this plane. The conformal invariance of H_2 is, of course, necessarily mutilated in the approximation process.

8. Generalization

The construction of the invariant in section 7 suggests that it may be useful to substitute equation (37) for $f^a{}_a$ into equation (47). If $f^a{}_a$ is the Laplacian of some scalar, $f^a{}_a = \nabla^2 G$, then one can perform two integrations by parts to re-express

$$\int \mathrm{d}A\, f^a{}_a (\mathbf{c}\cdot\mathbf{X}) = -\int \mathrm{d}A\, GK(\mathbf{c}\cdot\mathbf{n}). \tag{64}$$

It is now possible to peel off the space vector in equation (47) so that equation (48) is replaced by

$$\left(GK - \mathcal{H}^a{}_a\right)\mathbf{n} = \nabla_a \mathbf{F}^a. \tag{65}$$

A sufficient condition for conformal invariance is that $f^a{}_a = \nabla^2 G$, and $GK - \mathcal{H}^a{}_a$ is a Euler–Lagrange derivative. $\mathbf{F}^a$ is the corresponding stress tensor. This is clearly not the only way that conformal invariants arise. In fact, if the energy depends only on the intrinsic geometry, then $\mathcal{H}^{ab}$ vanishes and there are no solutions of this form. An interesting exercise would be to identify other energies with an $f^a{}_a$ which is the Laplacian of a scalar.

9. Discussions

In the statistical field theory of surfaces, conformally invariant Hamiltonians provide fixed points of the renormalization group flow. While the theory of two-dimensional surfaces is well understood [3], next to nothing is known about possible four-dimensional counterparts. The identification of the appropriate invariants is a small first step towards the formulation of such a theory. Before plunging into statistical field theory, however, there are questions of an elementary nature that should be addressed. What are the minima of the conformally invariant energy? Even without constraints, highly non-trivial vacua appear to be admitted. Is there a useful analogue of Willmore's conjecture [1]? The classification of solutions will involve topological selection rules beyond the scope of this paper. It would also be interesting to know if four-dimensional analogues of Goetz and Helfrich's egg carton geometries exist [20].

In the same way that the two-dimensional Willmore functional finds an application in relativistic field theory with the replacement of a Euclidean signature metric by a Lorentzian one, it is possible that the four-dimensional conformally invariant bending energy will find a role in braneworld cosmology [22]. In this context, the generalization of the conformally invariant energy (61) to accommodate a curved bulk should be straightforward.

Finally, it should be noted that our construction has been framed in the language of classical differential geometry. Its translation into the language of Cartan's exterior differential systems should be straightforward but useful, especially for addressing issues of a topological nature [23] (see also [24]).

Acknowledgments

I thank Riccardo Capovilla, Markus Deserno and Martin Müller for helpful comments. I am also grateful to Denjoe O' Connor for hospitality at DIAS Dublin where this work was begun. Partial support from CONACyT grants 44974-F and 47416 and DGAPA-PAPIIT grant IN114302 is acknowledged.

Appendix. Two-dimensional surfaces

When $D = 2$, the only polynomial in $\tilde{K}_{ab}$, with scaling dimension $d = -2$ and vanishing $\mathcal{H}^a{}_a$ is $\operatorname{tr} \tilde{K}^2 = \tilde{K}^{ab}\tilde{K}_{ab}$. The corresponding energy is the Willmore energy.

The Gauss–Bonnet topological invariant,

$$\int \mathrm{d}A \det K^a{}_b \tag{A.1}$$

is, of course, also a conformal invariant. For a two-dimensional surface $\det K^a{}_b = \mathcal{R}/2$. In fact, any quadratic invariant in extrinsic curvature is trivially also conformally invariant. This is because any scalar in the extrinsic curvature can be expressed as a linear combination of $\operatorname{tr} \tilde{K}^2 = \operatorname{tr} K^2 - K^2/D$ and $\mathcal{R} = K^2 - \operatorname{tr} K^2$, both of which give rise to conformal invariants when $D = 2$. Thus, in particular, the two quadratics in K_{ab}, K^2 and $\operatorname{tr} K^2$ also provide conformal invariants. This fact can be phrased in an alternative way. We note that both $\mathcal{H} = K^2$ and $\mathcal{H} = \operatorname{tr} K^2$ have $\mathcal{H}^a{}_a \propto K$ so that equation (49) is satisfied.

References

[1] Willmore T J 1982 *Total Curvature in Riemannian Geometry* (Chichester: Ellis Horwood)

[2] Konopelchenko B G 1996 *Stud. Appl. Math.* **96** 9
Taimanov I A 1997 *Solitons, Geometry, and Topology: On the Crossroad* (*Am. Math. Soc. Transl. Ser.* 2 179) (Providence, RI: American Mathematical Society) p 133

[3] Nelson D, Piran T and Weinberg S (ed) 1989 *Statistical Mechanics of Membranes and Surfaces* (*Proc. Jerusalem Winter School for Theoretical Physics* vol 5) (Singapore: World Scientific)

[4] David F, Ginsparg P and Zinn-Justin J (ed) 1994 *Fluctuating Geometries in Statistical Physics and Field Theory (Les Houches)* (Amsterdam: Elsevier)

[5] Canham P 1970 *J. Theor. Biol.* **26** 61
Helfrich W 1973 *Z. Naturf.* C **28** 693

[6] Polyakov A M 1987 *Gauge Fields and Strings* (Chur: Harwood Academic)

[7] Polyakov A M 1986 *Nucl. Phys.* B **286** 406
[8] Kleinert H 1986 *Phys. Lett.* A **114** 263
[9] Willmore T J 2000 *Ann. Glob. Anal. Geom.* **16** 255
[10] Arreaga G, Capovilla R and Guven J 2000 *Ann. Phys.* **279** 126
Capovilla R and Guven J 2002 *J. Phys. A: Math. Gen.* **35** 6233
[11] Guven J 2004 *J. Phys. A: Math. Gen.* **38** L313
[12] Spivak M 1979 *A Comprehensive Introduction to Differential Geometry* vol 4, 2nd edn (Boston, MA: Publish or Perish)
[13] Zhong-Can O Y and Helfrich W 1987 *Phys. Rev. Lett.* **59** 2486
Zhong-Can O Y and Helfrich W 1989 *Phys. Rev.* A **39** 5280
[14] Capovilla R, Guven J and Santiago J A 2003 *J. Phys. A: Math. Gen.* **36** 6281
[15] Müller M M, Deserno M and Guven J 2005 *Eur. Phys. Lett.* **69** 482
[16] Capovilla R, Cordero R and Guven J 1996 *Mod. Phys. Lett.* A **11** 2755
[17] Wald R M 1984 *General Relativity* (Chicage, IL: University of Chicago Press)
[18] Paneitz S 1983 A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds *Preprint*
[19] Kiessling M 2000 *Physica* A **279** 353
[20] Goetz R and Helfrich W 1996 *J. Phys. II Fr.* **6** 215
[21] Lipowsky R and Sackmann E 1995 Structure and dynamics of membranes *Handbook of Biological Physics* vol 1 and 2 (Amsterdam: Elsevier)
[22] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 3370
Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 4690
[23] Griffiths P *et al* 2003 *Exterior Differential Systems and Euler–Lagrange Partial Differential Equations (Chicago Lectures in Mathematics Series)*
[24] Tu Z C and Ou-Yang Z C 2004 *J. Phys. A: Math. Gen.* **37** 11407